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The Socle of an Alternative Ring

M. SLATER

The University, Bristol 8, England

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1. INTRODUCTION

The main facts about the minimal ideals and minimal right ideals of an associative ring are well known. In this paper we prove corresponding results for an alternative ring R. We make no restriction on the characteristic of R, but will often impose restrictions of semiprimeness type. (R is semiprime provided it has no ideal T such that $T \neq (0) = T^2$).

Throughout this paper "ring" will mean "alternative ring", and R will be a ring. We write $A \leqslant R$ ($A \leqslant_r R$; $A \leqslant_e R$) to denote that A is an ideal (right ideal, left ideal) of R. If A is a minimal element of the set $\{M : (0) \neq M \leqslant R\}$ (respectively, $\{M : (0) \neq M \leqslant_r R\}$), partially ordered by inclusion, we say that A is a *minimal ideal* (respectively, *minimal right ideal*) of R, and write $A \leqslant_m R$ (respectively, $A \leqslant_{mr} R$). Similarly for $A \leqslant_{me} R$.

In Section 2, we show that if $A \leqslant_m R$ then either $A^2 = (0)$ or A is simple; this result is due in part to Zhevlakov. In Section 3 we similarly characterize A if $A \leqslant_{mr} R$. Roughly, if $A^2 \neq (0)$, then A is a minimal right ideal of the ideal C it generates in R, and, in general, C is simple. We also show that A is of the form eR, where e is a nuclear idempotent.

In Section 4 we consider the *right socle* of $R : S_r(R) = \Sigma\{M : M \leqslant_{mr} R\}$. Under a suitable weak condition $S_r(R)$ is a two-sided ideal, and under a slightly stronger condition (weaker than semiprimeness of R) it coincides with the analogously defined left socle $S_e(R)$. In this situation we define the socle of R, $S(R)$, to be $S_r(R) = S_e(R)$. A structure theorem is proved for $S_r(R)$ which (like the results for minimal ideals and minimal right ideals) is particularly informative if R is purely alternative (free of nuclear ideals): that is, in some sense at the opposite extreme from being associative. We also show that $S_r(R)$ annihilates the Smiley radical $M(R)$, thus justifying Baer's name "antiradical" for $S_r(R)$.

One result of Section 4 is that if $A \leqslant R$ and R is semiprime, then $S(A) = A \cap S(R)$. In Section 5 we consider when a corresponding result holds for $A \leqslant_r R$. A closely related question is the following: Suppose $A \leqslant R$ or $A \leqslant_r R$. What natural conditions will guarantee that the collection of minimal ideals (or right ideals) of A coincides with the collection of those minimal ideals (right ideals) of R which lie in A? This question also receives a satisfactory answer, partly in Section 5, and partly in Section 3.

In Section 6 we consider the operator S which takes a ring R onto its socle $S(R)$, and two other operators, D and U, which arise naturally in the course of our investigation. We give a complete set of defining relations for the semigroup of operators generated by the set $\{D, U, S\}$; a typical relation, for example, is $SS = S$.

2. Minimal Ideals[1]

In this section we prove the following result:

Theorem A. *Suppose $A \leqslant_m R$ and $A^2 \neq (0)$. Then A is simple.*

The proof falls into two halves. First, this theorem has been proved by Zhevlakov (Lemma 4 of [15]) under the restriction $2A \neq (0)$. We rework his proof, partly for the reader's convenience, and partly because Zhevlakov's presentation is unnecessarily cumbersome. Next, we give an independent proof, valid under the restriction $3A \neq (0)$. Since $2A = (0)$ and $3A = (0)$ together imply $A^2 = (0)$, the proof is then complete.

Lemma 2.1. *If $A \leqslant_m R$ and $A^2 \neq (0)$, then*

(a) $A^2 = A$;

(b) *If $t \in A$ and $tA = (0)$ or $At = (0)$, then $t = 0$;*

(c) *If $mA \neq (0)$ (m an integer), then $mA = A$;*

(d) *If $mA \neq (0)$ (m an integer), then A is free of m-torsion.*

Proof. (a) If $A \leqslant R$ then also $A^2 \leqslant R$, and $A^2 \subseteq A$.

(b) If $l(A) = \{t \in A : tA = (0)\}$, then it is easily verified that $l(A) \subseteq A$ and $l(A) \leqslant R$. If $l(A) = A$ then $A^2 = (0)$. So $l(A) = (0)$. Similarly $r(A) = (0)$.

(c) Clearly $mA \subseteq A$ and $mA \leqslant R$.

(d) Set $T = \{a \in A : ma = 0\}$. Then $T \subseteq A$ and $T \leqslant R$. If $T = A$ then $mA = (0)$. So $T = (0)$. Thus A is free of m-torsion, in the sense that for $a \in A$, $ma = 0$ implies $a = 0$.

[1] See also Section 7.

LEMMA 2.2. *Suppose* $B \leqslant A \leqslant R$, *and set* $C = AB \cdot A + A \cdot BA$. *If* $(B, A, R) \subseteq C$, *then* $C \leqslant R$.

Proof. For $b \in B$; $a_1, a_2 \in A$; $r \in R$, we have, for example,

$$\begin{aligned} r \cdot (a_1 b) a_2 &= r(a_1 b) \cdot a_2 && - (r, a_1 b, a_2) \\ &= (r a_1)\, b \cdot a_2 - (r, a_1, b) a_2 && - (b', a_2, r) \\ &= a' b \cdot a_2 \quad + c a_2 && + c' (c, c' \in C) \\ &\equiv 0 \qquad (\mathrm{mod}\ C). \end{aligned}$$

Thus $C \leqslant_e R$, and by symmetry also $C \leqslant_r R$.

LEMMA 2.3. (Zhevlakov). *Every element of* $8(R^2)^2$ *can be written as a finite sum* $\Sigma \pm r_i^2$, *with* $r_i \in R$.

As noted by Zhevlakov in Lemma 3 of [15], this is an easy improvement on a consequence of Lemma 10 and the special case $n = 2$ of Lemma 11 in [14]. For convenience we give a direct proof.

Proof. Let R_1 be the linear span of all squares in R;

$$R_2 = R_1 + RR_1 + R_1 R; \quad R_3 = R_2 + RR_2 + R_2 R.$$

For $p, q, r, s \in R$, clearly $pq + qp \equiv 0 \ (\mathrm{mod}\ R_1)$,

whence

$$pq \cdot r + pr \cdot q = p(qr + rq) \equiv 0 \qquad (\mathrm{mod}\ R_2),$$

and

$$q \cdot pr + p \cdot qr = (qp + pq) r \equiv 0 \qquad (\mathrm{mod}\ R_2).$$

So

$$pq \cdot r \equiv -pr \cdot q \equiv q \cdot pr \equiv -p \cdot qr \qquad (\mathrm{mod}\ R_2).$$

Hence, we have, mod R_3, $pq \cdot rs \equiv -p \cdot q(rs) \equiv p \cdot (qr)s \equiv -p(qr) \cdot s \equiv (pq) r \cdot s \equiv -pq \cdot rs$. Thus, $2(pq)(rs) \equiv 0 \ (\mathrm{mod}\ R_3)$, so that $2(R^2)^2 \subseteq R_3$.

Next, for given $a, b \in R$ we have

$$\begin{aligned} 2a^2 b &= (a^2 + b)^2 - (a^2)^2 - b^2 + (ab + a)^2 - (ab)^2 - (ba + a)^2 + (ba)^2 \\ &\equiv 0 \qquad (\mathrm{mod}\ R_1), \end{aligned}$$

and $2ba^2 \equiv 0 \ (\mathrm{mod}\ R_1)$ similarly.

Thus $2R_2 \subseteq R_1$, whence

$$2R_3 = 2R_2 + 2R_2 \cdot R + R \cdot 2R_2 \subseteq R_1 + R_1 R + RR_1 = R_2.$$

So $8(R^2)^2 = 4.2(R^2)^2 \subseteq 4R_3 = 2.2R_3 \subseteq 2R_2 \subseteq R_1$, the desired conclusion.

PROPOSITION 2.4 (Zhevlakov). *Theorem A holds if* $2A \neq (0)$.

Proof. By 2.1a $A^2 = A$, and by 2.1c $2A = A$. Hence, $A = 8(A^2)^2$. Now let $B \leqslant A$ be given, and set $C = AB \cdot A + A \cdot BA$. By 2.3 every element of (B, A, R) is a sum of terms of the form (b, a^2, r). But $(b, a^2, r) = (ar + ra, b, a) \equiv 0 \pmod{C}$. So $(B, A, R) \subseteq C$, and by 2.2 $C \leqslant R$.

If $C = (0)$ then $AB \cdot A = (0)$, whence by 2.1(b) $AB = (0)$ and $B = (0)$. Otherwise, $C = A$, since $C \subseteq A$. But then $A = C \subseteq B \subseteq A$, yielding $B = A$. So A is simple, as required.

2.5. In what follows we write $N(R)$ for the nucleus of R, and $Z(R)$ for its center (for definitions see e.g., [9], Section 2). If $A \leqslant_r R$ we say that R is A-semiprime provided for $T \leqslant_r R$, $T^2 = (0)$ implies $T \cap A = (0)$. Equivalently, A contains no trivial right ideal of R. R is itself *semiprime* provided $T \leqslant R$ and $T^2 = (0)$ implies $T = (0)$. R is semiprime if and only if R is A-semiprime for every $A \leqslant_r R$. See [11], Section 3.

Lemma 2.5. *Suppose $A \leqslant_r R$, and R is A-semiprime. Then*

(a) $N(A) = A \cap N(R)$,

(b) *If $A \leqslant R$, then $Z(A) = A \cap Z(R)$.*

(c) *If $A \leqslant R$, then A is semiprime.*

(d) *Either* $3A \subseteq N(R)$ *or* $A \cap Z(R) \neq (0)$.

For proofs see [11], Theorems A, B, C, and Corollary 7.7, respectively.

Corollary 2.6. *If $A \leqslant_{mr} R$ and $A^2 \neq (0)$, then the conclusions of* 2.5 *hold.*

Proposition 2.7. Theorem A *holds if* $3A \neq (0)$.

Proof. By 2.5(c), A is semiprime. Let $B \leqslant A$ be given. By 2.5(d) applied to A, we have $3B \subseteq N(A)$ or $B \cap Z(A) \neq (0)$. But now by 2.5(a, b) applied to R, we deduce that $3B \subseteq N(R)$ or $B \cap Z(R) \neq (0)$. We consider these possibilities separately.

(i) $3B \subseteq N(R)$. Then $(0) = (3B, R, R) = 3(B, R, R)$, and since $(B, R, R) \subseteq A$, we deduce from 2.1(d) that $(B, R, R) = (0)$. In particular $(B, A, R) = (0)$. As in the proof of (2.4), we can now deduce that $B = (0)$ or $B = A$.

(ii) $B \cap Z(R) \neq (0)$. If $0 \neq b \in B \cap Z(R)$, it is easily verified that for $C = bA \subseteq B$, $C \leqslant R$. By 2.1(b) $C \neq (0)$. So $C = A$, and $A = C \subseteq B \subseteq A$ yields $B = A$.

Thus in all cases $B = A$ or (0), and A is indeed simple.

2.8. It is easy to extract some further information on a minimal ideal A

of R. Let $D_0 = D(R)$ be the associator ideal of R, and $U_0 = U(R)$ its maximum nuclear ideal. It is known that $D_0U_0 = U_0D_0 = (0)$ (see [10], Section 3). Then we have

THEOREM B. *Suppose $A \leqslant_m R$ and $A^2 \neq (0)$. Then exactly one of the following holds:*

(a) $A \subseteq U_0$. *In this case A is a simple associative ring.*

(b) $A \subseteq D_0$. *In this case A is a Cayley-Dickson algebra; $A = eR$ for e a central idempotent of R, and R is expressible as an ideal direct sum $R = A \oplus A'$.*

Proof. By minimality of A, exactly one of $A \subseteq D_0$; $A \cap D_0 = (0)$. If the latter, then from $(A, R, R) \subseteq A \cap D_0 = (0)$ we deduce $A \subseteq U_0$. This with Theorem A gives case (a).

Suppose now $A \subseteq D_0$. If A is associative, then $A = N(A) = A \cap N(R)$ by 2.6(a), whence $A \subseteq N(R)$, so $A \subseteq U_0$. But then $A \subseteq D_0 \cap U_0$, and $A^2 \subseteq D_0U_0 = (0)$, a contradiction.

By Theorem A we now know that A is simple but not associative. But then ([6], [9], [12]) A is a Cayley-Dickson algebra. If e is the unity of A, let $R = \Sigma R_{ij}$ be the corresponding Pierce decomposition of R. Then $R_{01} = R_{01}e \subseteq A \subseteq R_{11}$, yielding $R_{01} = (0) = R_{01}$ similarly. Thus $R = R_{11} \oplus R_{00}$, an ideal direct sum. Finally $A \subseteq R_{11} = eR \subseteq A$ yields $A = R_{11}$, and we set $R_{00} = A'$.

Recall now that R is *purely alternative* provided $U_0 = (0)$ (see [10], Section 4). Then we have the

COROLLARY 2.8. *Suppose R is semiprime and purely alternative. Then any minimal ideal is as in Theorem* B(b).

In particular, every minimal ideal is a direct summand. It is striking how much more we can say in this case than in the "opposite" case, when R is associative.

2.9. We can apply the results of this section to the theory of subdirectly irreducible rings. A ring R (not necessarily alternative) is said to be *subdirectly irreducible* provided the intersection M of its nonzero ideals is nonzero. (This will be so, if and only if, in any representation $R = \Sigma_s R_\gamma$ of R as a subdirect sum, at least one projection $\pi_\gamma : R \to R_\gamma$ has zero kerl, so that $R_\gamma \simeq R$). M is called the *heart* of R.

PROPOSITION 2.10. *Suppose R is subdirectly irreducible with heart M, and $M^2 \neq (0)$. Then exactly one of*

(a) *R is associative;*

(b) *$R = M$ is a Cayley-Dickson algebra.*

Proof. If $D_0 \neq (0)$ and $U_0 \neq (0)$, then $M \subseteq D_0 \cap U_0$, whence $M^2 \subseteq D_0 U_0 = (0)$, which is false. Thus, $D_0 = (0)$ or $U_0 = (0)$.

If $D_0 = (0)$ then R is associative, and we have case (a). If $U_0 = (0)$ then by Theorem B(b) $R = M \oplus M'$, and M is a Cayley-Dickson algebra. Since $M' \not\supseteq M$, we must have $M' = (0)$, and $R = M$. This gives case (b).

COROLLARY 2.11. *Suppose R is prime and has a minimal ideal M. Then exactly one of*

(a) *R is associative;*

(b) *$R = M$ is a Cayley-Dickson algebra.*

Proof. It is an easy exercise that a ring R (not necessarily alternative) is subdirectly irreducible with heart M such that $M^2 \neq (0)$ if and only if R is prime and has a minimal ideal M.

Note 2.12. Proposition 2.10 improves a result [8] of Kleinfeld, who obtains the same conclusion from the stronger hypothesis that M is not nil. As was pointed out in Kleinfeld's paper, subdirect irreducibility of R is not by itself enough to force the conclusion of 2.10.

3. MINIMAL RIGHT IDEALS

In this section we investigate the analog of Theorem B for the case where A is a minimal *right* ideal of R.

THEOREM C. *Suppose $A \leqslant_{mr} R$, C is the ideal of R generated by A, and $C^2 \neq (0)$. Then exactly one of the following holds:*

(a) *$C \subseteq U_0$. Then $A \leqslant_{mr} C$ and $C = CA$. If further $r(C) = (0)$, then C is simple.*

(b) *$C \subseteq D_0$. Then $A = C$ is a Cayley-Dickson algebra.*

Proof. Let T be the ideal of R generated by (A, A, R). Then $T \subseteq A$ (see [7], Lemmas 1 and 2). Since A is minimal, exactly one of $T = A$; $T = (0)$. We consider these cases separately.

(i) $T = A$. Then $A = C$, and clearly $A \leqslant_m R$. Since $A = T \subseteq D_0$, we are in the position of Theorem B(b), and thus conclusion (b) holds.

(ii) $T = (0)$. Note first that $A^2 \neq (0)$, or by [11], Lemma 3.3 we would have $C^2 = (0)$. Now by 2.6(a) we deduce from $(A, A, R) = (0)$ that $A = N(A) \subseteq N(R) = N$, say. Since $AR \subseteq A \subseteq N$ and since for any R we have $(R, N) \subseteq N$ (see [10], 2.6), we also have $RA \subseteq N$. But $C = A + RA$

(e.g., see [11], 2.16). Thus $C \subseteq N$. So $C \subseteq U_0$. We thus have the first assertion of (a).

Now let $E \leqslant_r C$ with $E \subseteq A$. Then, since $C \subseteq N$, $EC \subseteq A$ and $EC \leqslant_r R$. If $EC \neq (0)$, then $A = EC \subseteq E \subseteq A$ yields $E = A$. If $EC = (0)$, then $E \subseteq l(C)$. But $l(C)$ [for notation see proof of 2.1(b)] is an ideal of R. Clearly $l(C) \cap A \neq A$, since $A^2 \neq (0)$. So $l(C) \cap A = (0)$. Then

$$E = E \cap A \subseteq l(C) \cap A = (0).$$

We have thus shown that $A \leqslant_{mr} C$.

Next, since $A \subseteq N$, $A^2 \leqslant_r R$. Also $(0) \neq A^2 \subseteq A$, so that $A^2 = A$. Using $A \subseteq N$ we see that $CA \leqslant R$. Also $A = A^2 \subseteq CA$. Now C is the smallest ideal of R containing A. So $C \subseteq CA$, whence $C = CA$.

Suppose now $T \leqslant C$. By minimality of A, $T \cap A = A$ or (0). If $T \cap A = A$, then $A \subseteq T$ implies $C = CA \subseteq CT \subseteq T \subseteq C$, and $T = C$. If $T \cap A = (0)$, then $AT = (0)$, whence $CT = (CA)T = (0)$. Our extra condition now yields $T = (0)$.

Note 3.1. An extra condition is essential if C is to be simple. Thus let R be the algebra over any field F spanned by $\{a, t\}$, and with basis products $aa = a$, $ta = t$, $at = tt = 0$. If $A = Fa$, then $C = R = CA$, $A \leqslant_{mr} R$, but C is not simple. Here $0 \neq t \in r(C)$.

Note 3.2. If R is semiprime, then the two qualifications in the statement of Theorem B [that $C^2 \neq (0)$ and that $r(C) \neq (0)$] become superfluous.

PROPOSITION 3.3. *Suppose $A \leqslant_{mr} R$ and $A^2 \neq (0)$. Then*

(a) *If A generates the ideal C of R, then $A \leqslant_{mr} C$;*

(b) $A^2 = A$;

(c) $A = eR$ *for e a nuclear idempotent of R.*

Proof. (a) and (b). If (a) of Theorem C holds, we have seen that $A \leqslant_{mr} C$ and that $A^2 = A$. If (b) holds, both conclusions are trivial, since $A = C$ is an ideal of R, and, as a Cayley-Dickson algebra, $A \leqslant_{mr} A$.

(c) In case (a), $A \leqslant_{mr} C$ with C associative and $A^2 \neq (0)$. So by associative theory (e.g., [5], p. 57) we have $A = eC$ for $e \in C$ an idempotent. Then $e \in A$, whence $A = eC \subseteq eR \subseteq A$, yielding $A = eR$, and $e \in C$ implies $e \in N$. In case (b) $A = C = eR$ with e a central idempotent of R, by Theorem B(b).

Notes 3.4. We can obtain (a) of 3.3 without appeal to the known structure of simple alternative rings (that they are Cayley-Dickson algebras or associative). For, even without this information, we can show in the proof of Theorem C(b) that $A = C$ is simple but not associative (see also the proof of Theorem B(b)). But now by [7], Theorem 1, C has no proper right ideals, so that $A = C \leqslant_{mr} C$.

3.3(b) is not trivial, since it is not clear that in general the square of a right ideal is again a right ideal. (However, I know of no example where this is false).

We can use 3.3(c) to obtain an analog to (2.10).

PROPOSITION 3.5. *If R has the minimum right ideal A with $A^2 \neq (0)$, then $R = A$ is a Cayley-Dickson algebra or an associative division algebra.*

Proof. By 3.3 $A = eR$. Then $R = eR + (1 - e)R$, and $eR \nsubseteq (1 - e)R$ implies $(1 - e)R = (0)$. So $A = R$, and R has no proper right ideals. Hence the conclusion.

3.6. By 2.1(a) and 3.3(b) if $M \leqslant_m R$ or $M \leqslant_{mr} R$, then M is either *idempotent* ($M^2 = M$) or *trivial* [$M^2 = (0)$]. In the former case we write $M \leqslant_{im} R$ or $M \leqslant_{imr} R$.

We now give a result connecting the idempotent minimal ideals (right ideals) of R with those of a given ideal A of R. Comparable results when $A \leqslant_r R$ are given in Section 5.

THEOREM D. *Suppose $A \leqslant R$. Then*

(a) $\{M : M \leqslant_{im} A\} = \{M : M \leqslant_{im} R \;\&\; M \subseteq A\}$;

(b) $\{M : M \leqslant_{imr} A\} = \{M : M \leqslant_{imr} R \;\&\; M \subseteq A\}$.

Proof. (a) Suppose $B \leqslant_{im} A$. For $b_1, b_2 \in B$, $r \in R$, we have

$$(r, b_1, b_2) = (b_2, r, b_1) = a_1 b_1 - b_2 a_2 = b' \in B.$$

So

$$r \cdot b_1 b_2 = r b_1 \cdot b_2 - (r, b_1, b_2) = a' b_2 - b' = b'' \in B.$$

Thus $RB = RB^2 \subseteq B$; and similarly $BR \subseteq B$. So $B \leqslant R$, whence clearly $B \leqslant_{im} R$ and $B \subseteq A$.

Suppose $B \leqslant_{im} R$. Then by Theorem A B is simple; i.e. $B \leqslant_{im} B$. If also $B \subseteq A$, then *a fortiori* $B \leqslant_{im} A$.

(b) Suppose $B \leqslant_{imr} A$. Then $B = eA$ for $e \in N(A)$, by 3.3(c). But now, for $a \in A$, $r \in R$, $(e, a, r) = (e^2, a, r) = (e, a, er + re) = (e, a, a') = 0$. Thus $(e, A, R) = (0)$, whence $eA \leqslant_r R$. So $B \leqslant_r R$, whence clearly $B \leqslant_{imr} R$ and $B \subseteq A$.

Suppose $B \leqslant_{imr} R$. Then $B \leqslant_{imr} C$, where C is the ideal of R generated by B, by 3.3(a). If also $B \subseteq A$, we have $C \subseteq A$, whence *a fortiori* $B \leqslant_{imr} A$.

COROLLARY 3.7. *Suppose $A \leqslant R$ and R is A-semiprime. Then*

(a) $\{M : M \leqslant_m A\} = \{M : M \leqslant_m R \;\&\; M \subseteq A\}$;

(b) $\{M : M \leqslant_{mr} A\} = \{M : M \leqslant_{mr} R \;\&\; M \subseteq A\}$.

Proof. If $M \leqslant_m R$ or $M \leqslant_{mr} R$ and $M \subseteq A$, then the condition gives $M^2 = M$. Also by 2.5(c) $M \leqslant_m A$ or $M \leqslant_{mr} A$ implies $M^2 = M$. The result thus follows from Theorem D.

We will make essential use of 3.7(b) in the next section.

3.8. Since the conclusion of 3.7 refers only to *minimal* ideals and right ideals, it is reasonable to wonder whether a hypothesis of the same type can be used. Specifically, is it possible in (a) to assume merely that every *minimal* ideal of R contained in A is not trivial, and in (b) similarly for minimal right ideals ? In the light of Theorem D, we can rephrase this question as follows:

QUERY 3.8. *Given* $A \leqslant R$.

(a) *If* $M \leqslant_m R$ *[and* $M \subseteq A$*] implies* $M \leqslant_{im} R$, *is the same true of* A ?

(b) *If* $M \leqslant_{mr} R$ *[and* $M \subseteq A$*] implies* $M \leqslant_{imr} R$, *is the same true of* A ?

The answer to this query does not seem clear even when R is associative. See also 4.5 below.

4. THE SOCLE

The notion and name of *socle* were first used, for associative rings, by Dieudonné [2].

DEFINITION 4.1. The *right socle of* R, $S_r(R)$, is the sum of all the minimal right ideals of R. The *left socle* $S_\ell(R)$ is defined analogously.

We adopt the usual convention for the sum of an empty collection; thus if R has no minimal right ideals then $S_r(R) = (0)$.

4.2. If R is associative, it is well known that $S_r(R) \leqslant R$ (see [2], Proposition 1, or [1], Section 4). For arbitrary (alternative) R this is an open question. We prove it under a condition which is weaker than semiprimeness and also is vacuous for associative rings.

PROPOSITION 4.2. *Suppose whenever* $M \leqslant_{mr} R$ and $M \subseteq D_0$ *we have* $M^2 \neq (0)$. *Then* $S_r(R) \leqslant R$.

Proof. Suppose $A \leqslant_{mr} R$. If $A \not\subseteq D_0$ then, as in the proof of Theorem B, $A \subseteq U_0$. Thus $S_r = P_r + Q_r$ (with $S_r = S_r(R)$, etc.), where

$$P_r = \Sigma\{A : A \leqslant_{mr} R \;\&\; A \subseteq D_0\};$$
$$Q_r = \Sigma\{A : A \leqslant_{mr} R \;\&\; A \subseteq U_0\}.$$

By Theorem C(b) P_r is the sum of certain minimal two-sided ideals A of R which lie in D_0. Thus $P_r \leqslant R$.

Now suppose $A \leqslant_{mr} R$ and $A \subseteq U_0$, and let $r \in R$ be given. A and rA are right-R modules, and because $A \subseteq N(R)$, the mapping $a \rightarrow ra$ of A onto rA is a module homomorphism. Since A is irreducible, Schur's Lemma yields that $rA = (0)$ or is irreducible. Since also $rA \subseteq rU_0 \subseteq U_0$, we have $rA \subseteq Q_r$. Thus, $rQ_r \subseteq Q_r$, and $Q_r \leqslant R$. So $S_r = P_r + Q_r \leqslant R$.

4.3. For the rest of this paper we will assume the stronger condition that *all* minimal right ideals of R are idempotent. For convenience we give this condition a name:

DEFINITION 4.3. If $M \leqslant_{mr} R$ implies $M \leqslant_{imr} R$, we say that R has IMR (idempotent minimal right ideals). Similarly if $M \leqslant_{me} R$ implies $M \leqslant_{ime} R$, we say that R has IML.

PROPOSITION 4.4. *For* (b)–(h) *suppose R has* IMR. *Then*

(a) $A \leqslant R$ *and* $A \subseteq S_r$ *implies* $A = \Sigma\{M : M \leqslant_{mr} R \;\&\; M \subseteq A\}$.

(b) S_r *is semiprime.*

(c) $\{M : M \leqslant_{mr} R\} = \{M : M \leqslant_{mr} S_r\}$.

If $A \leqslant_r S_r$ *then*

(d) $A = \Sigma\{M : M \leqslant_{mr} R \;\&\; M \subseteq A\}$.

(e) $A \leqslant_r R$.

(f) $A^2 = A$.

(g) $S_r = A + B$, *an additive direct sum with* $B \leqslant_r R$.

(h) $A = S_r(A)$. *In particular*, $S_r(S_r(R)) = S_r(R)$.

Proof. (a) and (g). (a) is an easy application of the theory of modules: see [2], Section I, or [1], 4.1. If we assume (e), then (g) is also given by module theory (same Refs.).

(b) Since $M \leqslant_{mr} R$ implies $M^2 = M$, it follows from (a) that $A \leqslant_r R$ and $A \subseteq S_r$ implies $A^2 = A$. Thus R is S_r-semiprime. So by 2.5(c) and 4.2 S_r is semiprime.

(c) By (b) and 3.7(b), $\{M : M \leqslant_{mr} S_r\} = \{M : M \leqslant_{mr} R \;\&\; M \subseteq S_r\} = \{M : M \leqslant_{mr} R\}$ (by definition of S_r).

(d) By (c), $S_r = \Sigma\{M : M \leqslant_{mr} R\} = \Sigma\{M : M \leqslant_{mr} S_r\}$. Thus S_r, regarded as a right-S_r module, is a sum of irreducible submodules, so is completely reducible. Hence, so is any submodule A. That is, if $A \leqslant_r S_r$, then $A = \Sigma\{M : M \leqslant_{mr} S_r \;\&\; M \subseteq A\} = \Sigma\{M : M \leqslant_{mr} R \;\&\; M \subseteq A\}$ by (c)

(e) and (f). Immediate from (d).

(h) By (b) we may apply 3.7(b) to A and then to S_r to obtain

$$\begin{aligned} S_r(A) &= \Sigma\{M : M \leqslant_{mr} A\} = \Sigma\{M : M \leqslant_{mr} S_r \ \& \ M \subseteq A\} \\ &= \Sigma\{M : M \leqslant_{mr} R \ \& \ M \subseteq A\} = A \qquad \text{by (d).} \end{aligned}$$

Note 4.5. We may apply 4.4(a) to obtain a positive solution of 3.8(b) for the special case when $A \subseteq S_r(R)$. By 4.4(b) and 2.5(c) A is semiprime, and the result then follows by 3.7(b).

Similarly 3.8(a) has a positive solution if $A \subseteq \Sigma\{M : M \leqslant_m R\}$.

We now obtain a structure theorem for $S_r(R)$ under the hypothesis IMR. To clear the ground for this, we first give a quick discussion of infinite direct sums.

4.6. Suppose $\{C_\gamma : \gamma \in \Gamma\}$ is a collection of ideals of a ring R, and set $T = \Sigma\{C_\gamma : \gamma \in \Gamma\}$. If $S = \Sigma_e\{C_\gamma : \gamma \in \Gamma\}$ is the (external) direct sum of the rings C_γ (i.e., a certain ring of functions), then there is a homomorphism $\theta : S \to T$ given by $f\theta = \Sigma\{f(\gamma) : f(\gamma) \neq 0\}$. T is the *internal direct sum* of the C_γ, and we write $T = \Sigma_i\{C_\gamma : \gamma \in \Gamma\}$, provided θ is an isomorphism. A necessary and sufficient condition for this is that for each $\gamma \in \Gamma$ we have $C_\gamma \cap \Sigma\{C_\beta : \beta \neq \gamma\} = (0)$. Equivalently, any $0 \neq t \in T$ can be written uniquely as a finite sum Σt_γ, with $0 \neq t_\gamma \in C_\gamma$.

The next result is well known, and holds for any non-associative ring.

LEMMA 4.7. *Suppose $\{C_\gamma : \gamma \in \Gamma\}$ is a collection of ideals of R,* 1:1 *indexed by Γ, and every C_γ is a simple ring. Set $A = \Sigma\{C_\gamma : \gamma \in \Gamma\}$, and suppose $B \leqslant A$. Then*

(a) $A = \Sigma_i\{C_\gamma : \gamma \in \Gamma\}$.

(d) $B = \Sigma_i\{C_\gamma : \gamma \in \Delta = \Delta(B) \subseteq \Gamma\} = \Sigma_i\{C_\gamma : C_\gamma \subseteq B\}$.

(e) $B \leqslant R$.

(g) *If $E = \Sigma\{C_\epsilon : \epsilon \notin \Delta\}$, and $E' = \Sigma\{C_\gamma : C_\gamma \cap B = (0)\}$, then the sums defining E and E' are identical and direct (so that $E = E'$), and $A = B \oplus E$, a direct sum of ideals of R.*

Proof. (a) For given $\gamma \in \Gamma$ set $F_\gamma = \Sigma\{C_\beta : \beta \neq \gamma\}$, $G_\gamma = F_\gamma \cap C_\gamma$. Then $C_\gamma G_\gamma \subseteq C_\gamma F_\gamma \subseteq \Sigma\{C_\gamma C_\beta : \beta \neq \gamma\} \subseteq \Sigma\{C_\gamma \cap C_\beta : \beta \neq \gamma\} = (0)$. Since $G_\gamma \leqslant R$ and $G_\gamma \subseteq C_\gamma$ and C_γ is simple [in particular, $C_\gamma^2 \neq (0)$], it follows that $G_\gamma = (0)$. So by 4.6 the sum is direct.

(d) Let V be the ring of endomorphisms of $(A, +)$ generated by all right and left multiplications by elements of A. We can regard A as a right-V module. Then each C_γ is an irreducible submodule, and A is completely reducible. As in 4.4(d), we derive the first expression for B; it is the same as the second since for each $\gamma \in \Gamma$ we have $B \cap C_\gamma = C_\gamma$ or (0). The sum is direct by (a).

(e) is clear from (d).

(g) As in (d), the two sums are the same and are direct. It is obvious that $A = B \oplus E$, where E is defined by the first sum. B and E are ideals of R by (e).

We now return to alternative rings.

DEFINITION 4.8. Given R, we write $\mathscr{C} = \mathscr{C}(R)$ for the set of all those ideals C of R such that C is a simple ring having a minimal right ideal.

Note 4.9. Strictly we should write $\mathscr{C}_r$ for this set. However, in an obvious notation we have $\mathscr{C}_r = \mathscr{C}_e$. For if $C \in \mathscr{C}_r$, then $C \leqslant_m R$, whence by Theorem B C is associative or a Cayley-Dickson algebra. If the latter, then $C \leqslant_{me} C$ as well as $C \leqslant_{mr} C$, so that $C \in \mathscr{C}_e$. If C is associative with minimal right ideal, it also has a minimal left ideal (e.g., see [5], p. 65).

THEOREM E. *Suppose R has* IMR. *Then $S_r(R) = \Sigma\mathscr{C} = \Sigma_i\{C_\gamma : \gamma \in \Gamma\}$, where each C_γ is either a Cayley-Dickson algebra or isomorphic to a simple ring of linear transformations of finite rank on a vector space V_γ over as field D_γ.*

Proof. If $A \leqslant_{mr} R$, then by 3.3(a) $A \leqslant_{mr} C$, where $C = A + RA$. Since $A \subseteq S_r(R)$, $C \subseteq S_r(R) = S_r$, say. Now by 4.4(b) S_r is semiprime. So $r(C) = (0)$, and by Theorem C (both parts) C is simple. Also $A \leqslant_{mr} C$. Thus $A \subseteq C \in \mathscr{C}$. Hence, $S_r = \Sigma\{A : A \leqslant_{mr} R\} \subseteq \Sigma\{C : C \in \mathscr{C}\}$.

Conversely, suppose $C \in \mathscr{C}$, and let $A \leqslant_{mr} C$. Since C is semiprime, 3.7(b) yields $A \leqslant_{mr} R$. So $A \subseteq S_r$. Since C is simple and S_r is two-sided by 4.2, $C = A + CA \subseteq S_r$, whence $\Sigma\{C : C \in \mathscr{C}\} \subseteq S_r$. Thus $\Sigma\{C : C \in \mathscr{C}\} = S_r$.

Now let Γ be a 1:1 indexing set for $\mathscr{C}$. Since each C_γ is simple, 4.7 yields $S_r = \Sigma\{C_\gamma : \gamma \in \Gamma\} = \Sigma_i\{C_\gamma : \gamma \in \Gamma\}$. By Theorem B, each C_γ is either a Cayley-Dickson algebra or a simple associative ring having a minimal right ideal. In the latter case, the structure of C_γ was determined in [2], Theorem 4, and independently in [3], Theorem 9.

As an important corollary we have

THEOREM F. *Suppose R has* IMR *and* IML. *Then $S_r(R) = S_e(R)$.*

Proof. By Theorem D and its left-right analog, $S_r(R) = \Sigma\mathscr{C} = S_e(R)$.

DEFINITION 4.10. If R satisfies IMR and IML, we define the *socle of* R, $S(R)$, to be $S_r(R) = S_e(R)$.

Whenever in future we write $S(R)$, we imply that R satisfies these two conditions.

Notes 4.11. (a) The example of 3.1 shows that the hypothesis of Theorem E cannot be omitted.

(b) If R is the example of 3.1 with $F = Q$, the field of rationals, and if we now regard R as a ring without operators, then R satisfies IML but not IMR. Furthermore, $S_r(R) = R \neq (0) = S_e(R)$. Thus neither of the conditions IMR, IML can be omitted from Theorem F.

(c) The hypothesis of Theorem F is weaker than semiprimeness. For example, the zero ring on the group $(J, +)$ of integers is trivial, but satisfies IMR and IML. The condition on R commonly assumed in associative theory to ensure $S_r(R) = S_e(R)$ is semiprimeness; I do not know whether Theorem F for associative rings has previously appeared in the literature.

4.12. We now give the analog of 4.4 for *two-sided* ideals of S_r.

PROPOSITION 4.12. *Suppose R has* IMR *and $B \leqslant S_r(R)$. Then*

(d) $B = \Sigma_i\{\mathscr{C} : C \in \mathscr{C}\ \&\ C \subseteq B\}$.

(e) $B \leqslant R$.

(g) $S_r(R) = B \oplus E$, *a direct sum of ideals of R, with*

$$E = \Sigma_i\{C : C \in \mathscr{C}\ \&\ C \cap B = (0)\}.$$

Proof. We may apply 4.7, taking $A = S_r(R)$, since by Theorem E we have $S_r(R) = \Sigma\{C_\gamma : \gamma \in \Gamma\}$, with the C_γ satisfying the conditions of 4.7.

4.13. For our next results the hypothesis IMR suffices provided Query 3.8(b) has the answer *yes*. Since, however, 3.8 is open, we state the results under stronger hypotheses of semiprimeness type.

PROPOSITION 4.14. *Suppose $A \leqslant R$ and R is A-semiprime. Then $S(A) = A \cap S_r(R) = \Sigma\{C \in \mathscr{C} : C \subseteq A\}$.*

Proof. By 3.7(b)

$$\begin{aligned} S(A) &= \Sigma\{M : M \leqslant_{mr} A\} = \Sigma\{M : M \leqslant_{mr} R\ \&\ M \subseteq A\} \\ &= \Sigma\{M : M \leqslant_{mr} R\ \&\ M \subseteq A \cap S_r(R)\} = A \cap S_r(R) \end{aligned}$$

by 4.4(a).

Since $A \cap S_r(R) \leqslant S_r(R)$, 4.12 yields

$$A \cap S_r(R) = \Sigma\{C \in \mathscr{C} : C \subseteq A \cap S_r(R)\} = \Sigma\{C \in \mathscr{C} : C \subseteq A\}$$

by Theorem E.

PROPOSITION 4.15. *Suppose R is semiprime. Then $S(R) = P \oplus Q$, a direct sum of ideals of R, where $P = S(D_0)$ is the internal direct sum of all those* *ideals of R which are Cayley-Dickson algebras*; $Q = S(U_0)$ *is the internal direct sum of all those ideals of R which are simple associative rings having minimal right (and left) ideals.*

Proof. By Theorem B $\mathscr{C}$ is a disjoint union $\mathscr{C} = \mathscr{C}_1 + \mathscr{C}_2$ where $\mathscr{C}_1 = \{C \in \mathscr{C} : C \subseteq D_0\}$, and $\mathscr{C}_2 = \{C \in \mathscr{C} : C \subseteq U_0\}$. So $S(R) = \Sigma_i \mathscr{C} = \Sigma_i \mathscr{C}_1 \oplus \Sigma_i \mathscr{C}_2 = P \oplus Q$, say, where $P = \Sigma_i\{C \in \mathscr{C} : C \subseteq D_0\}$; $Q = \Sigma_i\{C \in \mathscr{C} : C \subseteq U_0\}$, by Theorem E and 4.7. By 4.14 $P = S(D_0)$ and $Q = S(U_0)$, and by Theorem B the C's in $\mathscr{C}_1$ and $\mathscr{C}_2$ respectively are those asserted.

4.16. We show finally that if R is semiprime then $S(R)$ annihilates the Smiley radical $M = M(R)$. The only facts we need concerning M are that $M \leqslant R$ and M contains no idempotent (see [13], Section 4). We therefore state the result in the following strong form:

PROPOSITION 4.16. *Suppose R has* IMR, *and $M \leqslant_r R$. If M contains no nonzero nuclear idempotent of R, then $M \cap S_r(R) = (0)$.*
[*In particular, $MS_r = (0)$, and if $M \leqslant R$ also $S_rM = (0)$*].

Proof. Set $T = M \cap S_r(R)$. Then $T \leqslant_r R$, and $T \subseteq S_r(R)$. So by 4.4(a) $T = \Sigma\{M : M \leqslant_{mr} R \;\&\; M \subseteq T\}$. Since any such M contains a nuclear idempotent of R by 3.3(c), the sum is empty and $T = (0)$.

Note 4.17. If R does not satisfy IMR, and M is the Smiley radical of R, we need not have $MS_r = (0)$, still less $M \cap S_r = (0)$, even when R is associative. Consider the example of 3.1. However, it is natural to ask whether we always have $S_rM = (0) = MS_e$. We prove this under the restriction used in 4.2.

PROPOSITION 4.18. *Suppose whenever $A \leqslant_{mr} R$ and $A \subseteq D_0$ we have $A^2 \neq (0)$. Let M be the Smiley radical of R. Then $S_rM = (0)$.*

Proof. Suppose $A \leqslant_{mr} R$ and $A \subseteq D_0$. Then by Theorem C A is a Cayley-Dickson algebra. Since M contains no idempotents, $A \cap M \neq A$, whence $A \cap M = (0)$. So $AM = (0)$.

If $A \leqslant_{mr} R$ and $A \not\subseteq D_0$ then $A \subseteq U_0$, as in the proof of Theorem B. We now imitate [1], Theorem 4.2. Given $a \in A$, $m \in M$, suppose $am \neq 0$. Then $am \in A$ generates the right ideal A of R, so that in particular $am \cdot (r + j) = a$ for some $r \in R$ and integer j. Thus $am' = a$ with $m' \in M$. Since m' is right quasi-regular, we deduce $a = 0$, whence $am = 0$ after all. So $AM = (0)$.

Thus $S_rM = (\Sigma A)M = \Sigma AM = (0)$.

5. The Socle of a Right Ideal

5.1. In this section we consider what analog there may be to 4.14 when we assume merely $A \leqslant_r R$. Suppose for example R is semiprime and $A \leqslant_r R$. A need not be semiprime, so that we cannot in general talk of $S(A)$. Although we can still consider $S_r(A)$, it is by no means always true that

$$S_r(A) = S_r(R) \cap A \; [=S(R) \cap A].$$

Consider the typical case where R is the $n \times n$ matrices over a field F ($n \geqslant 2$). If A is any proper right ideal of R, then A has right ideals which are not right ideals of R; $A \cap S(R) = A$, but $S_r(A) \subset A$. If we take $n = 2$, F the field Q of rationals, regard R as a ring without operators, and take A to consist of those matrices whose second row is zero, we find $S_r(A) = (0) \neq A = A \cap S(R)$.

Since even the best of associative rings are badly behaved in this respect, an extra condition that seems natural in this context is that R be *purely alternative*, i.e., that $U(R) = (0)$ (see 2.8). If we are concerned only with a given right ideal A of R, we can use a localized condition, just as for semiprimeness in 2.5. Specifically, given $A \leqslant_r R$, we say that R is *A-purely alternative* provided $V \leqslant_r R$ and $V \subseteq U(R)$ implies $V \cap A = (0)$.

Lemma 5.2. *Suppose $A \leqslant_r R$ and R is A-semiprime and A-purely alternative. Then A is semiprime and purely alternative.*

A proof may be found in [11], (6.5).

For later use we note the

Corollary 5.3. *Suppose $A \leqslant_r R$, R is A-semiprime, and $A \subseteq D(R)$. Then A has no nonzero associative right or left ideals.*

Proof. Suppose $V \leqslant_r R$ and $V \subseteq U_0$. *If* $W = V \cap A$, then $W^2 \subseteq U_0 D_0 = (0)$. So by A-semiprimeness $W = (0)$. Thus R is A-purely alternative. So by 5.2 A is semiprime and purely alternative. Now if $Y \leqslant_r A$ or $Y \leqslant_e A$ and Y is associative, then by 2.5(a) we have $Y = N(Y) = Y \cap N(A)$, whence $Y \subseteq N(A)$. So (as in the proof of (ii) in Theorem C) $Y \subseteq U(A) = (0)$. So $Y = (0)$, as required.

Theorem G. *Suppose $A \leqslant_r R$ and R is A-semiprime and A-purely alternative. Then the following are equal:*

(a) $\{M : M \leqslant_{mr} A\}$,

(b) $\{M : M \leqslant_{me} A\}$,

(c) $\{M : M \leqslant_{m} A\}$,

(a′) $\{M : M \leqslant_{mr} R \mathbin{\&} M \subseteq A\}$,

(b′) $\{M : M \leqslant_{me} R \mathbin{\&} M \subseteq A\}$,

(c′) $\{M : M \leqslant_{m} R \mathbin{\&} M \subseteq A\}$.

Furthermore, each such M is a Cayley-Dickson algebra.

Proof. If M is in (a) or (a′), or (b) or (b′), or (c) or (c′), then $M^2 \neq (0)$ since A is semiprime by 5.2, and $M \not\subseteq U(A)$ since A is purely alternative by 5.2. So M is a Cayley-Dickson algebra by Theorem C, its right-left analog, or Theorem B, respectively. In particular, M has no proper right or left ideals. Since $M \leqslant_r A$ or $M \leqslant_e A$, it follows that $M \in$ (a) or $M \in$ (b). But then, by Theorem C or its right-left analog, $M \in (c)$.

Now $M = D(M) \subseteq T$, where T is the ideal of R generated by (A, A, R). As in the proof of Theorem C, $T \subseteq A$, so by 2.5(c) T is semiprime. Also $M \leqslant_m M$ implies $M \leqslant_m T \leqslant R$. So 3.7 yields $M \leqslant R$. Since M has no proper right or left ideals, it is now clear that M lies in *all* the listed sets.

PROPOSITION 5.4. *Suppose $A \leqslant_r R$ and R is A-semiprime and A-purely alternative. Then $S(A) = \Sigma_i\{M : M \leqslant_m R \mathbin{\&} M \subseteq A\} = A \cap S_r(R)$. Also $S(A) \leqslant R$.*

Proof. Write S_0 for $S_r(R)$ and A_0 for $A \cap S_0$. Then $A_0 \subseteq S_0$ and $A_0 \leqslant_r R$. So by 4.4(a)

$$\begin{aligned} A_0 &= \Sigma\{M : M \leqslant_{mr} R \mathbin{\&} M \subseteq A_0\} = \Sigma\{M : M \leqslant_{mr} R \mathbin{\&} M \subseteq A\} \\ &= \Sigma\{M : M \leqslant_{mr} A\} \text{ (by Theorem G)} \\ &= S_r(A) = S(A) \text{ since by 5.2 } A \text{ is semiprime.} \end{aligned}$$

Next, $S(A) = \Sigma\{M : M \leqslant_{mr} A\} = \Sigma\{M : M \leqslant_m R \mathbin{\&} M \subseteq A\}$ by Theorem G. Each such M is a Cayley-Dickson algebra, again by Theorem G. So if Γ is a 1:1 indexing set for the collection of M's, $S(A) = \Sigma\{M_\gamma : \gamma \in \Gamma\} = \Sigma_i\{M_\gamma : \gamma \in \Gamma\}$ by 4.7. Since each $M_\gamma \leqslant R$, we also have $S(A) \leqslant R$.

Note 5.5. In the situation of 5.4, Theorem F also yields

$$S(A) = \Sigma_i\{M : M \leqslant_{me} R \mathbin{\&} M \subseteq A\},$$

whence $S(A) \subseteq S_e(R) \cap A$. It seems unlikely that equality necessarily holds, though I know of no counterexample.

5.6. In 5.1 we suggested that the failure in general of $S_r(A) = A \cap S(R)$ for $A \leqslant_r R$ and R semiprime was due to associativity. We can, however, also take the view that failure is due to the fact that A need not be semiprime. On this view the success of 5.4 should be attributed not so much to pure alternativity itself as to its consequence 5.2 that A *is* semiprime. We therefore now examine what happens when $A \leqslant_r R$ and A is semiprime.

PROPOSITION 5.7. *Suppose $A \leqslant_r R$ and A is semiprime. Then*

$$\{M : M \leqslant_{mr} A\} = \{M : M \leqslant_{mr} R \,\&\, M \subseteq A\}.$$

Proof. Suppose $B \leqslant_{mr} A$. Then by 3.3(c) $B = eA$ with $e \in N(A)$. Thus $e \in N(R)$ by 2.5(a), so $B \leqslant_r R$. Clearly then $B \leqslant_{mr} R$ and $B \subseteq A$.

Suppose conversely $B \leqslant_{mr} R$ and $B \subseteq A$. Then $B \leqslant_r A$, so $B^2 \neq (0)$. It follows by Theorem C that $B \subseteq X$ for $X = U_0$ or D_0. Set $A' = X \cap A$. Then $B \subseteq A'$, $B \leqslant_{mr} R$; $A' \subseteq X$. Also A' is semiprime since $A' \leqslant A$, in view of 2.5(c). If we can show that $B \leqslant_{mr} A'$, then clearly also $B \leqslant_{mr} A$. It thus suffices to consider the cases $A \subseteq D_0$ and $A \subseteq U_0$.

(i) $A \subseteq D_0$. By 5.3 A is purely alternative. So by Theorem G, $B \leqslant_{mr} A$.

(ii) $A \subseteq U_0$. If $(0) \neq C \subseteq B$ and $C \leqslant_r A$, set $E = CA \subseteq C$. Since A is semiprime, $E \neq (0)$. Since $A \subseteq U_0$, $E \leqslant_r R$. Also $E \subseteq B$. So $E = B$, whence also $C = B$. Thus, $B \leqslant_{mr} A$, as required.

PROPOSITION 5.8. *Suppose $A \leqslant_r R$ and A is semiprime. Then $S(A) = A \cap S_r(R)$.*

Proof. Set $A_0 = A \cap S_r(R)$. Then $A_0 \subseteq S_r(R)$ and $A_0 \leqslant_r R$. So by 4.4(a)

$$\begin{aligned} A_0 &= \Sigma\{M : M \leqslant_{mr} R \,\&\, M \subseteq A_0\} \\ &= \Sigma\{M : M \leqslant_{mr} R \,\&\, M \subseteq A\} \\ &= \Sigma\{M : M \leqslant_{mr} A\} \qquad \text{by 5.7} \\ &= S_r(A) = S(A), \text{ since } A \text{ is semiprime.} \end{aligned}$$

Note 5.9. By taking $A = Fa$ in the example of 3.1 we see that in the situation of 5.8 we need not have $S(A) \leqslant R$ (compare 5.4), or $S(A) \subseteq A \cap S_e(R)$ (compare 5.5).

Note 5.10. An examination of the proofs of 5.7 and 5.8 shows that the restriction that A be semiprime can be formally relaxed to the condition that R be A-semiprime and A have no total left zero-divisors. This at least indicates how much beyond A-semiprimeness is needed for 5.8. However, the improvement is illusory in view of the

Remark. Suppose $A \leqslant_r R$ and R is A semiprime. If $V \leqslant A$ and $V^2 = (0)$ then $VA = (0)$.

Proof. This is proved in the course of proving [11], 6.3.

6. The Operators D, U, S.

6.1. Let $\mathscr{A}$ be the class of all semiprime alternative rings, and let D, U, S be the functions which carry $R \in \mathscr{A}$ onto $D(R)$, $U(R)$, $S(R)$, respectively (see 4.10). The mappings carry $\mathscr{A}$ into $\mathscr{A}$ by 2.5(c). Let $\mathscr{T}$ be the semigroup with 1 of mappings of $\mathscr{A}$ which they generate. In this section, we characterize $\mathscr{T}$. We start with

Lemma 6.2. *If R is a direct sum of ideals R_γ, then $D(R)$ is the direct sum of the $D(R_\gamma)$, and $U(R)$ of the $U(R_\gamma)$.*

The proof, which goes through for any non-associative ring, is straightforward but tedious.

Theorem H. *Let $\mathscr{A}$ and $\mathscr{T}$ be as above. Then a complete set of defining relations for $\mathscr{T}$ in terms of D, U, S is*

$$UU = U \qquad SU = US \qquad DU = 0$$
$$SS = S \qquad SD = DSD = DS \qquad UD = 0.$$

Every $T \in \mathscr{T}$ can be written in exactly one of the following forms:

$$0 \qquad 1 \qquad U \qquad D^n \quad (n \geqslant 1)$$
$$S \qquad SU \qquad SD.$$

Proof. That $UU = U$ and $DU = 0$ is trivial. The assertion $UD = 0$ follows easily from 5.3. That $SS = S$ follows from 4.4(h). Next, given R, $S[D(R)]$ is a direct sum $\Sigma_i C_\gamma$ of Cayley-Dickson algebras. So by 6.2 $USD(R) = \Sigma_i U(C_\gamma) = (0)$. So $USD = 0$.

So by 4.15 and 6.2 $S(R) = SD(R) \oplus SU(R)$ whence

$$US(R) = USD(R) \oplus USU(R) = SU(R),$$

and we have $US = SU$.

Next, $SD(R) = \Sigma_i C_\gamma$ yields $DSD(R) = \Sigma_i D(C_\gamma) = \Sigma_i C_\gamma = SD(R)$. So $DSD = SD$. Hence also $DS(R) = DSD(R) \oplus DSU(R) = SD(R)$, and $DS = SD$. We have proved all the stated relations.

It is now clear that each $T \in \mathscr{T}$ can be written in at least one of the indicated forms. We produce a ring R on which the listed operations all differ in their action. This will show that each $T \in \mathscr{T}$ can be written in at *most* one of the given forms, and at the same time will show that our set of defining relations for $\mathscr{T}$ is complete.

Let F be a field, C the split Cayley-Dickson algebra over F; J the ring of integers, and E the split Cayley-Dickson algebra over the ring J. Set $R = F \oplus C \oplus J \oplus 2E$. Then

$$0(R) = (0) \qquad 1(R) = F \oplus C \oplus J \oplus 2E \qquad U(R) = F \oplus J$$
$$S(R) = F \oplus C \qquad SU(R) = F$$
$$D^n(R) = C \oplus 2^{3^n}E \quad (n \geqslant 1) \qquad SD(R) = C.$$

7. Another Proof of Theorem A [1]

In this section we give a proof of Theorem A (Section 2) which is valid without restriction on characteristic. The proof is based on ideas occurring in unpublished work by I. R. Hentzel.

7.1. Let V be any subset of a ring R. We define

$$V_0 = \text{the linear span of } V,$$
$$V_{n+1} = V_n + RV_n + V_nR.$$

Clearly the ideal V^* of R generated by V is

$$V^* = \cup \{V_n : n = 1, 2, \ldots\}.$$

Proposition 7.2. (Hentzel). *If $n \geqslant 1$, then*

$$V_{n+1} = V_n + V_nR \tag{a}$$
$$= V_n + RV_n\,. \tag{b}$$

Proof. For (a) it clearly suffices to proye $RV_n \subseteq V_n + V_nR$. Now, since $n \geqslant 1$,

$$\begin{aligned}
RV_n &= R(V_{n-1} + RV_{n-1} + V_{n-1}R),\\
&= RV_{n-1} + R \cdot RV_{n-1} + R \cdot V_{n-1}R,\\
&\subseteq RV_{n-1} + RR \cdot V_{n-1} + (R, R, V_{n-1}) + RV_{n-1} \cdot R + (R, V_{n-1}, R),\\
&= RV_{n-1} + RV_{n-1} \cdot R + (V_{n-1}, R, R) \qquad \text{(by alternativity)},\\
&\subseteq RV_{n-1} + RV_{n-1} \cdot R + V_{n-1}R \cdot R + V_{n-1} \cdot RR,\\
&\subseteq V_n + RV_{n-1} \cdot R + V_{n-1}R \cdot R + V_{n-1} \cdot R,\\
&= V_n + (RV_{n-1} + V_{n-1}R + V_{n-1})R,\\
&= V_n + V_nR,
\end{aligned}$$

as required. The proof of (b) is parallel.

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LEMMA 7.3. *Suppose* $V \leqslant A \leqslant R$. *Then*

(a) $(V + VR)A \subseteq (V + VR) \cap (V + RV)$,
$A(V + RV) \subseteq (V + VR) \cap (V + RV)$.

(b) $A(V + VR) \subseteq V + VR$,
$(V + RV)A \subseteq V + RV$.

(c) $VR \cdot A^2 \subseteq AV + AV$,
$A^2 \cdot RV \subseteq VA + AV$.

Proof. (a) If $v \in V$, $a \in A$, $r \in R$, then $(v, a, r) = va \cdot r - v \cdot ar = v'r + va' \in V + VR$. But also $(v, a, r) = -(r, a, v) = -ra \cdot v + r \cdot av = a''v + rv'' \in V + RV$. Thus $(V, A, R) \subseteq (V + VR) \cap (V + RV)$. So $vr \cdot a = v \cdot ra + (v, r, a) \in V + (V, A, R)$ gives the first part, and the second part is similar.

(b) $a \cdot vr = av \cdot r - (a, v, r) \in VR + (V, A, R) \subseteq VR + (V + VR)$ as above. This gives the first part, and the second is similar.

(c) Linearizing the Moufang identity $xr \cdot ax = x(ra) \cdot x$ we have $vr \cdot ab + br \cdot av = v(ra) \cdot b + b(ra) \cdot v$. Thus,

$$\begin{aligned} VR \cdot A^2 &\subseteq AR \cdot AV + V(RA) \cdot A + A(RA) \cdot V \\ &\subseteq AV + VA + AV, \end{aligned}$$

and $A^2 \cdot RV \subseteq VA + AV$ similarly.

COROLLARY 7.4. *If* $V \leqslant A \leqslant R$ *and* V_n *is as in* 7.1, *then* $V_n \leqslant A$.

Proof. An obvious induction, using 7.3(a) and (b).

LEMMA 7.5. *Suppose* $V \leqslant A \leqslant R$ *and* V_n *is as in* 7.1. *If* $A^2 = A$, *then* $V_nA + AV_n \subseteq V_1$ *for all* n.

Proof. By induction on n, the cases $n = 0$ and $n = 1$ being trivial by 7.4. Suppose we have it for given $n \geqslant 1$. Then

$$V_{n+1}A + AV_{n+1} = (V_n + V_nR)\, A^2 + A^2(V_n + RV_n)$$

by $A = A^2$ and 7.2,

$$\subseteq V_nA + (V_nA + AV_n) + AV_n + (V_nA + AV_n)$$

by 7.4 and 7.3(c),

$$\subseteq V_1$$

by inductive hypothesis. This completes the induction.

THEOREM A. *If* $A \leqslant_m R$ *and* $A^2 \neq (0)$, *then* A *is simple.*

Proof. Let V be a nonzero ideal of A, and V^* the ideal of R it generates. Then $V^* = A$, and by 2.1(a) $A^2 = A$. We are thus in the situation of 7.5, and we have

$$\begin{aligned} A = A^2 = V^*A &= \cup \{V_n : n = 1, 2, \ldots\} \cdot A \\ &= \cup \{V_n A : n = 1, 2, \ldots\} \\ &\subseteq V_1 \qquad \text{by 7.5.} \end{aligned}$$

Hence,

$$\begin{aligned} A = A^2 \subseteq V_1 A &= [(V + RV) + VR]A \\ &\subseteq (V + RV) + (V + RV) \qquad \text{by 7.3(b) and (a).} \end{aligned}$$

So $A = A^2 \subseteq A(V + RV) = A^2(V + RV) \subseteq V$ by 7.3(c). That is, $V = A$, and A has no proper ideals. So A is simple, as required.

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